Billiard arrays and finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules

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In this talk we will describe the notion of a **Billiard Array**.

This is a triangular array of one-dimensional subspaces of a finite-dimensional vector space, subject to several conditions that specify which sums are direct.

We will use Billiard Arrays to characterize the finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules, for $q$ not a root of unity.
Motivation: $U_q(\mathfrak{sl}_2)$ and its modules

In order to motivate things, we recall the quantum algebra $U_q(\mathfrak{sl}_2)$. We will use the equitable presentation.

Let $\mathbb{F}$ denote a field. Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity.
The definition of $U_q(\mathfrak{sl}_2)$

**Definition**

Let $U_q(\mathfrak{sl}_2)$ denote the associative $\mathbb{F}$-algebra with generators $x, y^{\pm 1}, z$ and relations $yy^{-1} = y^{-1}y = 1$,

\[
\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \\
\frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \\
\frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.
\]

The $x, y^{\pm 1}, z$ are called the **equitable generators** for $U_q(\mathfrak{sl}_2)$. 
The elements $\nu_x, \nu_y, \nu_z$

The defining relations for $U_q(\mathfrak{sl}_2)$ can be reformulated as follows:

$$q(1 - yz) = q^{-1}(1 - yz),$$
$$q(1 - zx) = q^{-1}(1 - xz),$$
$$q(1 - xy) = q^{-1}(1 - yx).$$

Denote these common values by $\nu_x, \nu_y, \nu_z$ respectively.
How $x, y, z$ are related to $\nu_x, \nu_y, \nu_z$

The $x, y, z$ are related to $\nu_x, \nu_y, \nu_z$ as follows:

\[
\begin{align*}
  x\nu_y &= q^2\nu_yx, \\
  y\nu_z &= q^2\nu_zy, \\
  z\nu_x &= q^2\nu_xz, \\
  y\nu_x &= q^{-2}\nu_xy, \\
  x\nu_z &= q^{-2}\nu_zx, \\
  z\nu_y &= q^{-2}\nu_yz.
\end{align*}
\]
For the rest of this talk, fix an integer $N \geq 1$.

Let $V$ denote an irreducible $U_q(\mathfrak{sl}_2)$-module, with dimension $N + 1$.

The $x, y, z$ act on $V$ as follows.

Each of $x, y, z$ is diagonalizable on $V$. Moreover there exists $\varepsilon \in \{1, -1\}$ such that for each of $x, y, z$ the eigenvalues on $V$ are $\{\varepsilon q^{N-2i}\}_{i=0}^{N}$. This ordering and its inversion will be called standard.

The scalar $\varepsilon$ is called the type of $V$.

Replacing $x, y, z$ by $\varepsilon x, \varepsilon y, \varepsilon z$ the type becomes 1.

From now on, assume that $V$ has type 1.
The $\nu_x, \nu_y, \nu_z$ act on $V$ as follows.

Each of $\nu_x, \nu_y, \nu_z$ is nilpotent on $V$.

Moreover, for $\rho \in \{x, y, z\}$ the subspace $\nu^i_\rho V$ has dimension $N - i + 1$ for $0 \leq i \leq N$, and $\nu^{N+1}_\rho V = 0$. 
Decompositions and flags

In order to clarify how $x, y, z$ and $\nu_x, \nu_y, \nu_z$ act on $V$, we use the following concepts.

By a **decomposition** of $V$ we mean a sequence $\{V_i\}_{i=0}^N$ of one-dimensional subspaces of $V$ whose direct sum is $V$.

**Example**

For each of $x, y, z$ the sequence of eigenspaces (in standard order) is a decomposition of $V$, said to be **standard**.
By a flag on $V$ we mean a sequence $\{U_i\}_{i=0}^N$ of subspaces for $V$ such that $U_{i-1} \subseteq U_i$ for $1 \leq i \leq N$ and $U_i$ has dimension $i + 1$ for $0 \leq i \leq N$.

**Example**

Each of

$$\{\nu_x^{N-i} V\}_{i=0}^N, \quad \{\nu_y^{N-i} V\}_{i=0}^N, \quad \{\nu_z^{N-i} V\}_{i=0}^N$$

is a flag on $V$, said to be **standard**.
From decompositions to flags

Given a decomposition \( \{ V_i \}_{i=0}^{N} \) of \( V \) we construct a flag on \( V \) as follows.

Define \( U_i = V_0 + \cdots + V_i \) for \( 0 \leq i \leq N \). Then the sequence \( \{ U_i \}_{i=0}^{N} \) is a flag on \( V \).

This flag is said to be **induced** by the decomposition \( \{ V_i \}_{i=0}^{N} \).
Let \( \{ U_i \}_{i=0}^N \) and \( \{ U'_i \}_{i=0}^N \) denote flags on \( V \).

These flags are called **opposite** whenever \( U_i \cap U'_j = 0 \) if \( i + j < N \) (\( 0 \leq i, j \leq N \)).

The flags \( \{ U_i \}_{i=0}^N \) and \( \{ U'_i \}_{i=0}^N \) are opposite if and only if there exists a decomposition \( \{ V_i \}_{i=0}^N \) of \( V \) that induces \( \{ U_i \}_{i=0}^N \) and whose inversion \( \{ V_{N-i} \}_{i=0}^N \) induces \( \{ U'_i \}_{i=0}^N \).

In this case \( V_i = U_i \cap U'_{N-i} \) for \( 0 \leq i \leq N \).

So we say that the decomposition \( \{ V_i \}_{i=0}^N \) is **induced** by the opposite flags \( \{ U_i \}_{i=0}^N \) and \( \{ U'_i \}_{i=0}^N \).
The standard flags and decompositions

**Theorem**

For our $U_q(sl_2)$-module $V$, the three standard flags are mutually opposite.

The standard flags are related to the standard decompositions in the following way.

**Theorem**

For our $U_q(sl_2)$-module $V$,

(i) each standard decomposition of $V$ induces a standard flag on $V$;

(ii) each ordered pair of distinct standard flags on $V$ induces a standard decomposition of $V$.
The above theorems suggest a problem in linear algebra.

Consider the three standard flags on our $U_q(\mathfrak{sl}_2)$-module $V$.

From these flags we can recover the standard decompositions of $V$, and from them the original $U_q(\mathfrak{sl}_2)$-module structure.

So these flags should be related in a special way, from a linear algebraic point of view.

The problem is to describe this relationship.

This is what we will do, for the rest of the talk.
Recall the natural numbers \( \mathbb{N} = \{0, 1, 2, \ldots \} \).

**Definition**

Let \( \Delta_N \) denote the set consisting of the three-tuples of natural numbers whose sum is \( N \). Thus

\[
\Delta_N = \{(r, s, t) \mid r, s, t \in \mathbb{N}, \ r + s + t = N\}.
\]
The set $\Delta_N$

We arrange the elements of $\Delta_N$ in a triangular array.

For $N = 4$, the array looks as follows after deleting all punctuation:

```
040
130 031
220 121 022
310 211 112 013
400 301 202 103 004
```

An element in $\Delta_N$ is called a **location**.
The lines in $\Delta_N$

In the above array, each horizontal row consists of the locations with the same middle coordinate.

Call the horizontal rows 2-lines.

The 1-lines and 3-lines are similarly defined.

By a line we mean a 1-line or 2-line or 3-line.
In the above array, each interior location is adjacent to six other locations.

By a 3-clique we mean a set of three mutually adjacent locations.

There are two kinds of 3-cliques: $\Delta$ (white) and $\nabla$ (black).
We now define a Billiard Array.

Let $V$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$.

**Definition**

By a **Billiard Array on** $V$ we mean a function $B$ that assigns to each location $\lambda \in \Delta_N$ a 1-dimensional subspace of $V$ (denoted $B_\lambda$) such that:

(i) for each line $L$ in $\Delta_N$ the sum $\sum_{\lambda \in L} B_\lambda$ is direct;

(ii) for each white 3-clique $C$ in $\Delta_N$ the sum $\sum_{\lambda \in C} B_\lambda$ is not direct.

We say that $B$ is **over** $\mathbb{F}$. We call $N$ the **diameter** of $B$. 

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**Billiard arrays and finite-dimensional irreducible $U_q(sl_2)$-modules**
Let $B$ denote a Billiard Array on $V$.

It turns out that the function $B$ is injective.

We view $B$ as an arrangement of one-dimensional subspaces of $V$ into a triangular array, with the subspace $B_\lambda$ at location $\lambda$ for all $\lambda \in \Delta_N$.

Thus the subspaces $B_\lambda$ are the “billiards” in the array.
Here is our plan for the rest of the talk:

(i) Classify the Billiard Arrays up to isomorphism.

(ii) Describe what the Billiard Arrays have to do with 3-tuples of mutually opposite flags.

(iii) Use Billiard Arrays to explain what is special about the three standard flags for a finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module of type 1.
Our next goal is to classify the Billiard Arrays up to isomorphism.

**Lemma**

Let \( \lambda, \mu, \nu \) denote locations in \( \Delta_N \) that form a white 3-clique. Then the subspace \( B_\lambda + B_\mu + B_\nu \) is equal to each of

\[
B_\lambda + B_\mu, \quad B_\mu + B_\nu, \quad B_\nu + B_\lambda.
\]

This subspace has dimension 2.

**Corollary**

Let \( \lambda, \mu, \nu \) denote locations in \( \Delta_N \) that form a white 3-clique. Then each of \( B_\lambda, B_\mu, B_\nu \) is contained in the sum of the other two.
Among the lines in $\Delta_N$, three are on the boundary.

**Lemma**

*Let $L$ denote a boundary line of $\Delta_N$. Then*

$$V = \sum_{\lambda \in L} B_\lambda \quad \text{(direct sum)}.$$
Shortly we will classify the Billiard Arrays up to isomorphism.

To prepare for this, we explain what isomorphism means in this context.

**Definition**

Let $V'$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$. Let $B'$ denote a Billiard Array on $V'$. By an **isomorphism of Billiard Arrays from $B$ to $B'$** we mean an $\mathbb{F}$-linear bijection $V \to V'$ that sends $B_\lambda \mapsto B'_\lambda$ for all $\lambda \in \Delta_N$. The Billiard Arrays $B$ and $B'$ are called **isomorphic** whenever there exists an isomorphism of Billiard Arrays from $B$ to $B'$. 
We now describe the notion of an affine brace.

**Definition**

Let $\lambda, \mu, \nu$ denote locations in $\Delta_N$ that form a white 3-clique. By an **affine brace** (or **abrace**) for this clique, we mean a set of vectors

$$u \in B_\lambda, \quad v \in B_\mu, \quad w \in B_\nu$$

that are not all zero, and $u + v + w = 0$. (In fact each of $u, v, w$ is nonzero).
Here is an example of an abrace.

**Example**

Let \( \lambda, \mu, \nu \) denote locations in \( \Delta_N \) that form a white 3-clique. Pick any nonzero vectors

\[
u \in B_\lambda, \quad v \in B_\mu, \quad w \in B_\nu.
\]

The vectors \( u, v, w \) are linearly dependent. So there exist scalars \( a, b, c \) in \( \mathbb{F} \), not all zero, such that \( au + bv + cw = 0 \). The vectors \( au, bv, cw \) form an abrace for the clique.
Affine braces have the following property.

**Lemma**

Let $\lambda, \mu, \nu$ denote locations in $\Delta_N$ that form a white 3-clique. Then each nonzero vector in $B_\lambda$ is contained in a unique abrace for this clique.
We have been discussing affine braces.

We now consider a variation on this concept, called a brace.

**Definition**

Let $\lambda, \mu$ denote adjacent locations in $\Delta_N$. Note that there exists a unique location $\nu \in \Delta_N$ such that $\lambda, \mu, \nu$ form a white 3-clique. We call $\nu$ the **completion** of the pair $\lambda, \mu$. 
The braces for a Billiard Array, cont.

Definition

Let $\lambda, \mu$ denote adjacent locations in $\Delta_N$. By a brace for $\lambda, \mu$ we mean a set of nonzero vectors $u \in B_{\lambda}, v \in B_{\mu}$ such that $u + v \in B_\nu$. Here $\nu$ denotes the completion of $\lambda, \mu$. 

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Braces have the following property.

**Lemma**

Let $\lambda, \mu$ denote adjacent locations in $\Delta_N$. Each nonzero vector in $B_\lambda$ is contained in a unique brace for $\lambda, \mu$. 
We now define some maps $\tilde{B}_{\lambda,\mu}$.

**Definition**

Let $\lambda, \mu$ denote adjacent locations in $\Delta_N$. We define an $\mathbb{F}$-linear map $\tilde{B}_{\lambda,\mu} : B_{\lambda} \to B_{\mu}$ as follows. This map sends each nonzero $u \in B_{\lambda}$ to the unique $v \in B_{\mu}$ such that $u, v$ is a brace for $\lambda, \mu$. 

Let $\lambda, \mu$ denote adjacent locations in $\Delta_N$.

We just defined an $\mathbb{F}$-linear map $\tilde{B}_{\lambda,\mu} : B_\lambda \to B_\mu$.

We now consider what happens when we compose the maps of this kind.
Lemma

Let $\lambda, \mu$ denote adjacent locations in $\Delta_N$. Then the maps $\tilde{B}_{\lambda, \mu} : B_\lambda \to B_\mu$ and $\tilde{B}_{\mu, \lambda} : B_\mu \to B_\lambda$ are inverses.
Lemma

Let $\lambda, \mu, \nu$ denote locations in $\Delta_N$ that form a white 3-clique. Then the composition around the clique:

$$B_\lambda \xrightarrow{\tilde{B}_{\lambda,\mu}} B_\mu \xrightarrow{\tilde{B}_{\mu,\nu}} B_\nu \xrightarrow{\tilde{B}_{\nu,\lambda}} B_\lambda$$

is equal to the identity map on $B_\lambda$. 
The maps $\tilde{B}_{\lambda,\mu}$, cont.

**Definition**

Let $\lambda, \mu, \nu$ denote locations in $\Delta_N$ that form a black 3-clique. Then the composition around the clique:

$$
\begin{align*}
B_\lambda \xrightarrow{\tilde{B}_{\lambda,\mu}} B_\mu \xrightarrow{\tilde{B}_{\mu,\nu}} B_\nu \xrightarrow{\tilde{B}_{\nu,\lambda}} B_\lambda
\end{align*}
$$

is a nonzero scalar multiple of the identity map on $B_\lambda$. The scalar is called the **clockwise $B$-value** (resp. **c.clockwise $B$-value**) of the clique whenever the sequence $\lambda, \mu, \nu$ runs clockwise (resp. c.clockwise) around the clique.
Lemma

For each black 3-clique in $\Delta_N$, its clockwise $B$-value and c.clockwise $B$-value are reciprocals.

Definition

For each black 3-clique in $\Delta_N$, by its $B$-value we mean the clockwise $B$-value.

We have now assigned a nonzero scalar value to each black 3-clique in $\Delta_N$. 
We define a function $\hat{B}$ on the set of black 3-cliques in $\Delta_N$. The function $\hat{B}$ sends each black 3-clique to its $B$-value.

We call $\hat{B}$ the **value function** for $B$. 
It is convenient to view $\hat{B}$ as a function on $\Delta_{N-2}$, as follows.

Pick $(r, s, t) \in \Delta_{N-2}$. Observe that the locations

$$(r, s + 1, t + 1), \quad (r + 1, s, t + 1), \quad (r + 1, s + 1, t)$$

are in $\Delta_N$ and form a black 3-clique.

The $B$-value of this 3-clique is equal to the image of $(r, s, t)$ under $\hat{B}$. 
We just defined the value function of a Billiard Array.

We will use these value functions to classify the Billiard Arrays up to isomorphism.

**Definition**

By a *value function* on $\Delta_N$ we mean a function $
\psi : \Delta_N \to \mathbb{F}\setminus\{0\}$. 

The classification of Billiard Arrays

We now classify the Billiard Arrays up to isomorphism.

Recall the Billiard Array $B$ and its value function $\hat{B}$.

**Theorem**

The map $B \mapsto \hat{B}$ induces a bijection between the following two sets:

(i) the isomorphism classes of Billiard Arrays over $\mathbb{F}$ that have diameter $N$;

(ii) the value functions on $\Delta_{N-2}$.
Our next goal is to describe what Billiard arrays have to do with 3-tuples of mutually opposite flags.

Until further notice let $V$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$. 
Totally opposite flags

Definition

Suppose we are given three flags on $V$, denoted $\{U_i\}_{i=0}^N$, $\{U'_i\}_{i=0}^N$, $\{U''_i\}_{i=0}^N$. These flags are said to be **totally opposite** whenever $U_{N-r} \cap U'_{N-s} \cap U''_{N-t} = 0$ for all integers $r, s, t$ ($0 \leq r, s, t \leq N$) such that $r + s + t > N$. 
Given three flags on $V$, the totally opposite condition is somewhat stronger than the mutually opposite condition.

This is explained on the next slide.
Lemma

Given three flags on $V$, denoted $\{U_i\}_{i=0}^N$, $\{U'_i\}_{i=0}^N$, $\{U''_i\}_{i=0}^N$. Then the following are equivalent:

(i) the flags $\{U_i\}_{i=0}^N$, $\{U'_i\}_{i=0}^N$, $\{U''_i\}_{i=0}^N$ are totally opposite;
(ii) for $0 \leq n \leq N$ the sequences

\[
\{U_i\}_{i=0}^{N-n}, \quad \{U_{N-n} \cap U'_{n+i}\}_{i=0}^{N-n}, \quad \{U_{N-n} \cap U''_{n+i}\}_{i=0}^{N-n}
\]

are mutually opposite flags on $U_{N-n}$. 

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Billiard arrays and finite-dimensional irreducible $U_q(sl_2)$-modules
We are going to show that the Billiard Arrays on $V$ are in bijection with the 3-tuples of totally opposite flags on $V$.

To get started, we show how to get a Billiard Array on $V$ from a 3-tuple of totally opposite flags on $V$. 

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Lemma

Suppose we are given three totally opposite flags on $V$, denoted $\{U_i\}_{i=0}^N$, $\{U'_i\}_{i=0}^N$, $\{U''_i\}_{i=0}^N$. For each location $\lambda = (r, s, t)$ in $\Delta_N$ define

$$B_\lambda = U_{N-r} \cap U'_{N-s} \cap U''_{N-t}.$$ 

Then the function $B$ on $\Delta_N$ that sends $\lambda \mapsto B_\lambda$ is a Billiard Array on $V$. 

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Billiard arrays and finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules
Consider the following two sets:

(i) the 3-tuples of totally opposite flags on $V$;
(ii) the Billiard Arrays on $V$.

In the previous lemma we described a function from (i) to (ii).

**Theorem**

The above function is a bijection.
Our next goal is to use Billiard Arrays to explain what is special about the three standard flags for a finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module of type 1.
The main theorem

Let $V$ denote a finite-dimensional irreducible $U_q(sl_2)$-module, with type 1 and dimension $\geq 2$. Then:

(i) the three standard flags on $V$ are totally opposite;

(ii) for the corresponding Billiard Array on $V$, the value of each black 3-clique is a constant $q^2$.  

In this talk, we first considered a finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module $V$ of type 1.

We defined three flags on $V$, called the standard flags.

We then introduced the notion of a Billiard Array on a vector space $V$.

We classified the Billiard Arrays up to isomorphism, using the notion of a value function.

We showed how the Billiard Arrays on $V$ are in bijection with the 3-tuples of totally opposite flags on $V$.

We showed that for the above $U_q(\mathfrak{sl}_2)$-module $V$, the three standard flags are totally opposite, and for the corresponding Billiard Array the value function is constant, taking the value $q^2$.

Thank you for your attention!

THE END

